Journal of Statistical Physics, Vol. 61, Nos. 1/2, 1990

Slowing Down of Retrieval in the Hopfield Model

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Received April 28, 1990

The variation of the convergence time τ, as a function of the storage capacity α is studied numerically for systems ranging in size from $N=1000$ to $N=16{,}000$ neurons. τ is found to increase like $\tau \sim \exp[-A(\alpha_c-\alpha)]$ as one nears the critical storage capacity $\alpha_c = 0.142 = 0.002$.

KEY WORDS: Hopfield model; neural networks; spin glasses; relaxation time.

Recently, the question of convergence time in the Hopfield model of neural networks has become of interest as one of the few remaining problems in the model still in need of a more complete understanding.[1,2] Previous work has shown that at fixed (nonzero) storage capacity α, the convergence time to a fixed point τ grows like $\tau \sim O(\ln N)$ (where N is the number of nodes in the network).[1,2] As $\alpha \to 0$, it can be shown analytically that $\tau \sim O(\ln \ln N)$.[7] In this paper we are concerned with the opposite limit, namely the limit as α approaches its critical value $\alpha_c = 0.142 \pm 0.002$ at fixed but large N.

In the closely related problem of spin-glasses,[3] the relaxation time τ is thought to diverge, as the temperature T goes to zero, like $\ln(\tau/\tau_0) \sim T^{-\beta}$, where β is a constant. As T approaches the critical temperature T_c from above, the relaxation time is thought to have either a power law, $\tau \sim (T-T_c)^{-\beta}$, or a Vogel–Fulcher, $\tau \sim \exp[A/(T-T_c)]$, behavior (see, e.g., ref. 4 for recent references). Since the spin-glass-like transition at α_c in the Hopfield model at zero temperature has been rigorously studied both

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0022-4715/90/1000-0501$06.00/0

analytically and numerically,[5,6] we have looked for similar behavior for the convergence time, τ. Instead, we find the convergence time to be consistent with $\tau \sim \exp[-A(N)(\alpha_c - \alpha)]$, which compares more favorably to the results for $T=0$ spin-glasses.[3] To see how this comes about, let us consider the problem in more detail.

The Hopfield model is defined for an N-spin Ising system by choosing $P=\alpha N$ patterns $\{\xi_i^\mu\}$ at random and "storing" them in a network by forming Hebbian couplings $\{J_{ij}\}$:

$$J_{ij} = \sum_\mu \xi_i^\mu \xi_j^\mu \tag{1}$$

The network is then allowed to update according to a zero-temperature, fully parallel dynamics:

$$S_i(t+1) = \operatorname{sgn}\left(\sum_{j \neq i} J_{ij} S_j(t)\right) \tag{2}$$

The system is started with a corrupted version of any of the stored patterns $\{\xi_i^\mu\}$. The corrupted pattern will evolve following the above dynamics. If the initial overlap $m(0) \equiv (1/N)\sum_j \xi_i^\mu S_j(0)$ is within the domain of attraction of the learned pattern, then either the network recognizes the stored pattern perfectly as the dynamics brings the corrupted patterns back to the learned pattern (for $\alpha = 0$); or it recognizes the pattern somewhat imperfectly as the dynamics brings it to a state with significant overlap with the stored pattern (for $0 < \alpha \leqslant 0.142 \pm 0.002$[5,6]). As mentioned above, this retrieval dynamics is expected to become slow near the phase transition driven by the storage capacity α. We study here the variation of the retrieval time for patterns with an initial overlap $m(0)=0.80$ and $m(0)=0.95$ as a function of α. These values of the overlap should be large enough that most of the initial states end up near the desired stored state as $\alpha \to \alpha_c$.

Our simulations are done on fully connected networks of $N=16{,}000$ and $N=1000$ neurons. The computer algorithm used is the multineuron coding algorithm discussed in ref. 6, to which we refer the interested reader for details.

After starting the system in an initial state with one of the above overlaps, the time needed for convergence to a fixed point was measured. In all cases, the resulting final overlap $m(\infty)$ was in the range $m(\infty) \geqslant 0.975$. Figure 1 shows three histograms for the distribution of the convergence

[3] Preliminary results reported at the Workshop on Neural Networks, CTS, Hc, Bangalore, March 1990.

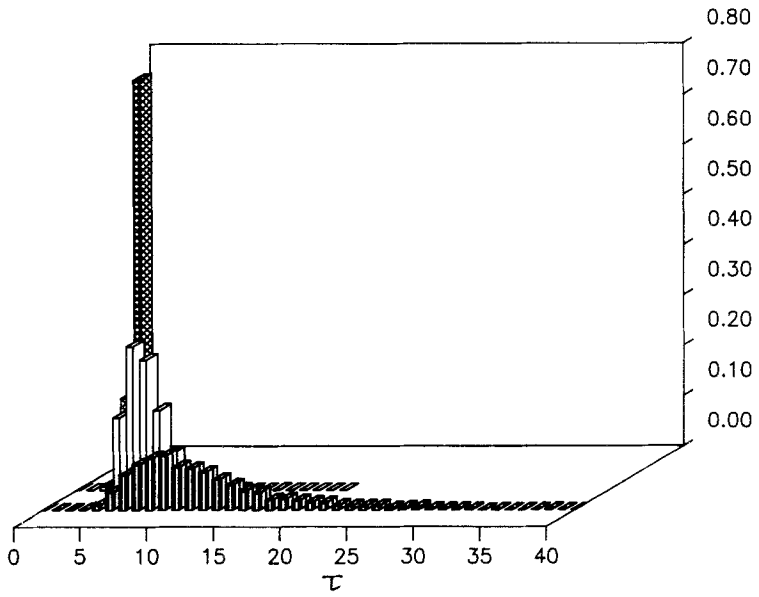

Fig. 1. Histograms for the distribution of convergence time for $m(0)=0.95$. From back to front; $\alpha=0.70$, 0.110, 0.130. The data are taken at $N=16{,}000$.

time taken at $\alpha=0.070$, $\alpha=0.110$, and $\alpha=0.130$ with $m(0)=0.95$ in each case. Each histogram represents 4000 initial starting configurations chosen from 20 sets of random patterns. Note that the average relaxation time $\langle\tau\rangle$ increases as α increases toward α_c, while at the same time the distribution spreads out. Since the functional form of the distribution of relaxation times is not known and previous experience shows that it may indeed have a strong N dependence,[2] we concentrate our analysis on the average relaxation time.

Figure 2 shows a plot of the average convergence time $\langle\tau\rangle$ as a function of α for both $m(0)=0.80$ and $m(0)=0.95$. From this graph a Vogel–Fulcher type of behavior is ruled out and fits to a power law were not convincing. In the insert is shown a plot of $\ln\langle\tau\rangle$ vs. $\alpha_c-\alpha$. This indicates that the average convergence time behaves like $\tau\sim\exp[-A(N)(\alpha_c-\alpha)^{\beta}]$, with β of order unity. The actual value of β may have an N and $m(0)$ dependence [we find $\beta\approx 0.60$ to be a good fit for $N=16{,}000$ and $m(0)=0.95$], but the data are not accurate enough to give a clear indication. From previous results for spin-glasses, such a functional form is, as explained earlier, not totally unexpected.

In summary, we find that the average convergence time increases

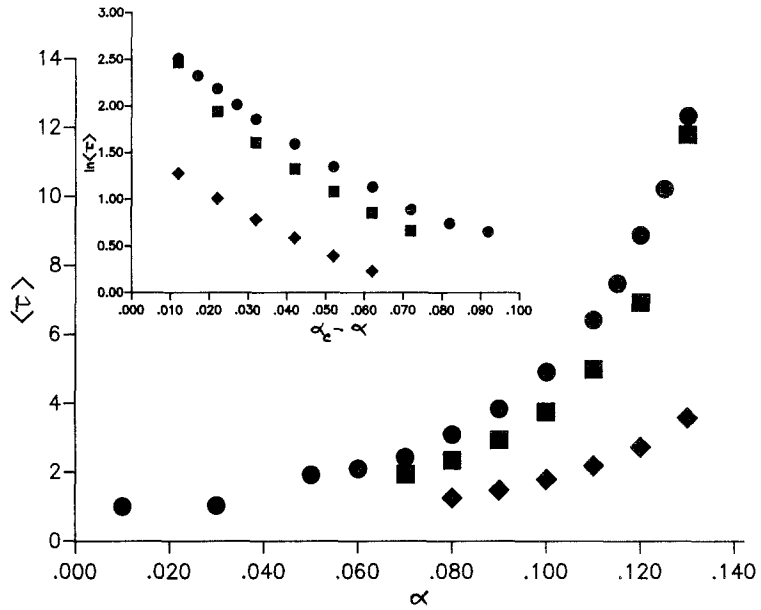

Fig. 2. Plot of average convergence time as a function of α for $m(0) = 0.80$ (circles) and $m(0) = 0.95$ (squares) at $N = 16{,}000$, and for $m(0) = 0.95$ (diamonds) at $N = 1000$. The insert shows the best fit as discussed in the text.

exponentially as $\alpha \to \alpha_c$. These results should be interpreted with caution until such time as they can be confirmed on larger systems, which experiences[2] indicates may be on the order of $N \sim 10^6$.

ACKNOWLEDGMENTS

We would like to thank C. Dasgupta and D. Stauffer for many useful discussions.

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Communicated by D. Stauffer.